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## ISOMETRIC EMBEDDINGS OF KÄHLER-RICCI SOLITONS IN THE COMPLEX PROJECTIVE SPACE

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**ABSTRACT.** We prove that a compact complex manifold endowed with a non-trivial Kähler-Ricci soliton cannot be isometrically embedded in the Fubini-Study complex projective space as a complete intersection.

### INTRODUCTION

A Kähler metric  $g$  on a complex manifold  $M$  is said to be a Kähler Ricci soliton if there exists a holomorphic vector field  $V$  on  $M$  such that

$$(0.1) \quad \text{Ric}(g) = \lambda g + \mathcal{L}_V g,$$

where  $\lambda$  is a real constant. Kähler Ricci solitons have been extensively studied in recent years mainly because they provide self-similar solutions to the Kähler Ricci flow which was introduced as a mean for finding Kähler-Einstein metrics. Kähler Ricci solitons are indeed a generalization of Kähler-Einstein metrics (taking  $V = 0$  in (0.1) we get the Einstein equation) but they are alternative to them because the presence of a Kähler Ricci soliton with nontrivial  $V$  is an obstruction to the existence of a Kähler-Einstein metric on a compact complex manifold with positive first Chern class (The Futaki invariant with respect to the real part of  $V$  is nonzero). In fact it is a deep result proved by Tian and Zhu [10] that a compact Fano manifold can admit at most one Kähler Ricci soliton, including trivial ones.

The first nontrivial examples of compact Kähler Ricci solitons were found by Koiso: in [5] he proved the existence of a KRS on any Fano manifold admitting a cohomogeneity one action of a compact semisimple Lie group of isometries with two complex singular orbits. After that, Wang and Zhu [11] proved the existence of KRS on any compact toric Fano manifold and this result was later generalized in [8] to toric bundles over generalized flag manifolds. Since all compact KRS are Fano and can be holomorphically embedded in the complex projective space  $\mathbb{CP}^m$ , it is natural to ask whether a Kähler Ricci soliton may be induced by the Fubini-Study metric of  $\mathbb{CP}^m$ .

In this note we prove the following negative result. Recall that a smooth codimension  $r$  subvariety of  $\mathbb{CP}^m$  is a complete intersection if its ideal is generated by  $r$  elements or equivalently if it may be described as the transverse intersection of  $r$  algebraic hypersurfaces.

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**Theorem 0.1.** *Let  $M$  be a closed complex submanifold of  $\mathbb{CP}^m$  such that the metric induced on  $M$  by the Fubini-Study metric  $\omega_{FS}$  is a Kähler-Ricci soliton. If  $M$  is a complete intersection then the Kähler-Ricci soliton is trivial and  $M$  is a linear subspace or a smooth quadric subvariety of some linear subspace.*

Our result may be thought as a generalization of the main theorem of [3] where the classification of Kähler Einstein manifolds isometrically embedded in  $(\mathbb{CP}^n, \omega_{FS})$  as complete intersections is given. For general smooth subvarieties, beside the homogeneous case of flag manifolds (see [9] for the classification), no example of positive Kähler Einstein metric induced by  $\omega_{FS}$  is known. On the other hand a Kähler Einstein submanifold of  $(\mathbb{CP}^n, \omega_{FS})$  has necessarily positive scalar curvature by a result of Hulin [4].

## 1. PROOF OF THE THEOREM

**1.1. Kähler Ricci solitons.** Let  $M$  be a complex manifold and denote by  $J$  its complex structure. Rephrasing (0.1) in terms of 2-forms, a Kähler Ricci soliton on  $M$  is a Kähler metric  $g$  whose associated, Ricci and Kähler form  $\rho$  and  $\omega = g(J\cdot, \cdot)$  respectively satisfy

$$(1.1) \quad \rho = \lambda\omega + \mathcal{L}_V\omega$$

for some holomorphic vector field  $V = X - iJX$ , where  $J$  is the complex structure. We will say that the Kähler Ricci soliton is *trivial* if  $V = 0$ , i.e.  $(M, g)$  is Kähler-Einstein.

Note that  $\mathcal{L}_X J = 0$  because  $V$  is holomorphic and equation (1.1) implies that  $\mathcal{L}_{JX}\omega = 0$ , i.e.  $JX$  preserves  $\omega$ , hence  $g$  because it also preserves  $J$ . Note also that (1.1) implies

$$(1.2) \quad \rho = \lambda\omega + \mathcal{L}_X\omega.$$

The fact that  $\nabla X$  is  $g$ -self adjoint means that the 1-form dual to  $X$  is closed; since a KRS may exist only on Fano manifolds and these are simply connected (Kobayashi's theorem), we see that  $X$  is the gradient with respect to  $g$  of some smooth function  $f$ . We will indicate  $\nabla f := \text{grad}_g(f)$ . This implies that

$$\mathcal{L}_X\omega = \mathcal{L}_{\nabla f}\omega = d\iota_{\nabla f}\omega = d\iota_{J\nabla f}\omega = dd^c f.$$

Recalling<sup>1</sup> that  $\partial = \frac{1}{2}(d + id^c)$  and  $\bar{\partial} = \frac{1}{2}(d - id^c)$ , equation (1.2) turns out to be equivalent to

$$(1.3) \quad \rho = \lambda\omega + 2i\partial\bar{\partial}f.$$

Indeed the previous computation shows also that the function  $f$  indeed admits another useful interpretation. Since  $\iota_{JX}\omega = -df$  the function  $f$  is, up to a constant multiple, a *moment map* for the infinitesimal action of the Killing vector field  $JX$  on  $M$ , or more precisely it is the projection along  $JX$  of a moment map  $\mu$  for the Hamiltonian action of  $\text{Iso}(M, g)$  on  $M$ . (Recall that since  $M$  is simply connected every symplectic action on  $M$  is Hamiltonian)

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<sup>1</sup>We are using the convention according to which  $d^c h(Y) = Jdh(Y) = dh(-JY)$ .

**1.2. proof of the theorem.** Let  $n$  be the complex dimension of  $M$  and  $r = m - n$  the codimension. Denote also by  $i : M \rightarrow \mathbb{CP}^m$  the inclusion and simply by  $\omega$  the restriction  $i^*\omega_{FS}$ . By hypothesis  $\omega$  satisfies (1.3) where  $f$  is the potential of the holomorphic vector field  $X = \nabla f$ . We suppose that  $M$  is embedded in  $\mathbb{CP}^m$  as a complete intersection. Namely  $M$  is assumed to admit  $r$  homogeneous polynomials  $P_1, P_2, \dots, P_r$  on  $\mathbb{C}^{m+1}$  which define  $M$  as their zero locus and generate the ideal associated to  $M$ .

It is a direct consequence of the adjunction formula that the canonical line bundle  $K_M = \Lambda^{n,0}M$  of  $M$  is the restriction of a line bundle on  $\mathbb{CP}^m$ , more precisely

$$K_M = i^*\mathcal{O}(d - m - 1)$$

where  $d = \sum_{j=1}^r \deg P_j$ . Since the Chern class of  $K_M$  is represented by  $\frac{1}{2\pi}$  times the Ricci form, the constant  $\lambda$  in (1.3) is forced to be equal to  $m + 1 - d > 0$ .

It is well known Hermitian metrics  $h$  on  $K_M^*$  correspond bijectively to positive volumes (nowhere vanishing real  $2n$ -forms)  $v$  of  $M$ , the correspondence being given by

$$\langle v, (-2)^m (\sqrt{-1})^{m^2} x \wedge \bar{x} \rangle = h(x, x)$$

for  $x \in K_M^*$ . Let  $V$  be the volume of  $M$  corresponding to the fibre metric on  $K_M^*$  whose Chern curvature form is exactly  $(m + 1 - d)\omega$ . In [3] (proposition 2) it is computed explicitly the real positive function  $\phi$  such that  $\omega^n = \phi V$  in the case where  $M$  is a complete intersection. More precisely, recalling that the Chern curvature form of the fibre metric induced by  $\omega$  on  $K_M^*$  is exactly the Ricci form  $\rho$  (see [1], p.82), we have the following

**Proposition 1.1** (Hano [3]). Let  $M$  be a complete intersection in  $\mathbb{CP}^m$  defined by the polynomials  $P_1, \dots, P_r$ . Denote by  $d = \sum_i \deg P_i$  and by  $\rho$  the Ricci form of the metric  $\omega$  induced by  $\omega_{FS}$ . Then we have

$$(1.4) \quad \rho = (m + 1 - d)\omega + i\partial\bar{\partial}\log \phi, \quad \text{with} \quad \phi = \frac{\|dP_1 \wedge dP_2 \wedge \dots \wedge dP_r\|^2}{\|z\|^{2(d-r)}}.$$

Here  $\phi$  is expressed in terms of unitary homogeneous coordinates of  $\mathbb{CP}^m$  and  $\|dP_1 \wedge dP_2 \wedge \dots \wedge dP_r\|^2 = \sum |P_{\lambda_1 \dots \lambda_r}|^2$  where  $dP_1 \wedge dP_2 \wedge \dots \wedge dP_r = \sum P_{\lambda_1 \dots \lambda_r} dz_{\lambda_1} \wedge \dots \wedge dz_{\lambda_r}$ . Note also that the previous expression of  $\phi$  is invariant under any unitary coordinate transformation.

Combining (1.4) with the Kähler-Ricci soliton equation we get

$$\partial\bar{\partial}\log \phi = 2\partial\bar{\partial}f.$$

so that

$$\phi = C \cdot e^{2f}$$

for some constant  $C \in \mathbb{R}$ . Now the key fact is that we can find an explicit expression also for  $f$  in terms of homogeneous coordinates of  $\mathbb{CP}^m$ . Indeed, as already remarked,  $f$  is a moment map for the action of the 1-parameter group of isometries generated by  $JX$  and this enables us to write it down in suitable coordinates. To start with, by a famous result of Calabi [2] the Killing vector field  $JX$  can be extended to a Killing vector field of  $(\mathbb{CP}^m, \omega_{FS})$  so that with respect to an appropriate system of unitary homogeneous coordinates it can be written in diagonal form  $\text{diag}(i\lambda_0, \dots, i\lambda_m)$  as an element of  $\mathfrak{su}(m + 1)$ .

Thus a moment map for the Hamiltonian action of the 1-parameter group  $\{\exp tJX\}$  on  $\mathbb{CP}^m$  is

$$\mu_{JX} = \frac{1}{2} \frac{\sum_{j=0}^m \lambda_j |z_j|^2}{\sum_{j=0}^m |z_j|^2},$$

and  $f$  is nothing but the restriction of  $\mu_{JX}$  to  $M$ . So there exists a constant  $C \in \mathbb{R}$  such that on  $M$  one has

$$(1.5) \quad \frac{\|dP_1 \wedge dP_2 \wedge \cdots \wedge dP_r\|^2}{\|z\|^{2(d-r)}} = Ce^{\frac{\sum \lambda_j |z_j|^2}{\sum |z_j|^2}}.$$

We claim that (1.5) holds if and only if  $f(z, \bar{z})$  is constant. Let  $p$  and  $q$  be any two points of  $M$ . Since  $M$  is Fano, by a Theorem of Kollár Miyaoka and Mori [6] there exists a rational curve passing through  $p$  and  $q$ , say  $F : \mathbb{CP}^1 \rightarrow M \subseteq \mathbb{CP}^m$  defined by  $F([s : t]) = [F_0(s, t) : \cdots : F_m(s, t)]$  where the functions  $F_m(s, t)$  are homogeneous polynomials of degree  $\delta$  in  $s$  and  $t$ .

Evaluating (1.5) at  $F(\mathbb{CP}^1)$  we get

$$(1.6) \quad \frac{\|dP_1(F([s : t])) \wedge \cdots \wedge dP_r(F([s : t]))\|^2}{(\sum_j |F_j(s, t)|^2)^{(d-r)}} = Ce^{\frac{\sum \lambda_j |F_j(s, t)|^2}{\sum |F_j(s, t)|^2}}$$

for every  $[s : t] \in \mathbb{CP}^1$  and this is clearly impossible unless  $f$  is constant on  $F(\mathbb{CP}^1)$ , otherwise the right hand side of (1.6) would not be a rational function of  $s$  and  $t$ . Since  $p$  and  $q$  are arbitrary,  $f$  must be constant on all of  $M$ : this means that  $X = \nabla f$  vanishes and the Kähler-Ricci soliton is trivial, i.e.  $\omega$  is Kähler-Einstein. According to Hano [3] this happens only if  $M$  is a linear subspace or it is a smooth quadric subvariety of some linear subspace.

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